

Electrodynamic Radiation Reaction and General Relativity

Carlos Kozameh¹, Ezra T. Newman², Raul Ortega³ Gilberto Silva-Ortigoza⁴

¹FaMaF, Univ. of Cordoba,
Cordoba, Argentina
kozameh@famaf.unc.edu.ar

²Dept of Physics and Astronomy,
Univ. of Pittsburgh,
Pittsburgh, PA 15260, USA
newman@pitt.edu

³FACEN, Univ. of Catamarca,
Catamarca, Argentina
nikkoortega@yahoo.com.ar

⁴Facultad de Ciencias Físico Matemáticas
de la Universidad Autónoma de Puebla,
Apartado Postal 1152, 72001,
Puebla, Pue., México
gsilva@cfm.buap.mx

March 27, 2008

Abstract

We argue that the well-known problem of the instabilities associated with the self-forces (radiation reaction forces) in classical electrodynamics are possibly stabilized by the introduction of gravitational forces via general relativity.

1 Introduction

The problems and difficulties associated with the motion of charged particles interacting with both an external electromagnetic field and its own self-field have not been resolved even after over a century of investigation[1, 2, 3, 4, 5]. The problems arise in many different contexts: the difficulties in giving appropriate initial conditions, infinite self-energy problems, model building, Lorentz invariance difficulties and perhaps the most serious, the instabilities (or pre-acceleration) in the solutions to the equations of motion.

The best known equations of motion, coming from a point structureless particle, are the Abraham-Lorentz equations and the relativist generalization the Abraham-Lorentz-Dirac equations. They are given, respectively, by[3]

$$m \vec{\dot{v}} = q \vec{E} + q \vec{B} \times \vec{v} + \frac{2q^2}{3c^3} \vec{\dot{v}}, \quad (1)$$

and[2]

$$m \dot{v}^a = q F^{ab} v_b + \frac{2q^2}{3c^3} (\ddot{v}^a + \frac{1}{c^2} v^a \dot{v}^b \dot{v}_b), \quad (2)$$

where F^{ab} (or \vec{E}, \vec{B}) are an external field and are derived by a variety of means, but always with severe approximations.

It is generally acknowledged that there are fundamental difficulties with this issue. And there seems to be a variety of different reasons, explanations and suggested remedies. They range from: quantum theory is the resolution to the approximations leading to these equations are wrong or even that there is no real problem. The author, J. D. Jackson, summarizes the situation in his well known graduate text[1] as:

“The difficulties presented by this problem touch one of the most fundamental aspects of physics, the nature of the elementary particle. Although partial solutions, workable within limited areas, can be given, the basic problem remains unsolved. One might hope that the transition from classical to quantum-mechanical treatments would remove the difficulties. While there is still hope that this may eventually occur, the present quantum-mechanical discussions are beset with even more elaborate troubles than the classical ones. It is one of the triumphs of comparatively recent years (~1948 - 1950) that the concepts of Lorentz covariance and gauge invariance were exploited sufficiently cleverly to circumvent these difficulties in quantum electrodynamics and so allow the calculation of very small radiative effects to extremely high precision, in full agreement with experiment. From a fundamental point of view, however, the difficulties remain.”

The purpose of this note is to describe a totally new point of view towards this problem. This new view is completely classical, with no reliance on quantum theory. It however does rely heavily on the Einstein-Maxwell equations of general relativity.

The basic situation that we address is to first consider an arbitrary compact gravitating-electromagnetic system which is taken to be the particle whose motion we want to describe. The system is given by local mass and charge densities and currents. There are no external fields acting on it. It is an isolated system with arbitrary internal degrees of freedom. The program is to solve the Einstein-Maxwell equations in the future null asymptotic region and, from the asymptotic field (the asymptotic Weyl and Maxwell tensors), determine a center of mass and center of charge and their laws of motion. Aside from conditions that the total charge Q be non-vanishing and the important requirement that the centers of mass and charge should coincide, there is no further model building.

Since our detailed calculations, which were done in the language of the spin-coefficient formalism, are long and complicated and have appeared elsewhere[6], we will just summarize the ideas and results. Basically the calculations are done 2^{nd} order in deviations from Reissner-Nordstrom. In the spherical harmonic expansions, with frequent use of Clebsch-Gordon products, only terms up to the $l = 2$ harmonics are kept.

2 The Complex Center of charge: Its Identification

The basic starting idea in this work is essentially simple. It is in the generalizations and implementations where difficulties arise.

Starting in Minkowski space in a given Lorentzian frame with spatial origin, the electric dipole moment \vec{D}_E is calculated from an integral over the (localized) charge distribution. If there is a shift, \vec{R} , in the origin, the dipole transforms as

$$\vec{D}_E^* = \vec{D}_E - Q \vec{R}. \quad (3)$$

If \vec{D}_E is time dependent, we obtain the center of charge world-line by taking $\vec{D}_E^* = 0$, i.e., from $\vec{R} = \vec{D}_E/Q$. It is this idea that we want to generalize and extend to gravitational fields.

First, however, we want to discuss other dipole issues in flat-space. Starting on the time-like world-line at the spatial origin, we construct the family of future directed light-cones, \mathfrak{C}_0 , and investigate behavior of the Maxwell field in the limit as null infinity is approached, i.e., Penrose's \mathfrak{J}^+ . Using the null tetrad formalism and where the Maxwell pair $(\vec{E}, \vec{B}$ or $F_{ab})$ is replaced by the complex vector $\vec{E} + i\vec{B}$, or more accurately by their tetrad components, (ϕ_0, ϕ_1, ϕ_2) , with[7]

$$\begin{aligned} \phi_0 &= F_{ab} l^a m^b \\ \phi_1 &= \frac{1}{2} F_{ab} (l^a n^b + m^a \bar{m}^b) \\ \phi_2 &= F_{ab} \bar{m}^a n^b. \end{aligned} \quad (4)$$

The asymptotic (peeling) behavior of these fields for a compact source is given by

$$\begin{aligned} \phi_0 &= \frac{\phi_0^0}{r^3} + O(r^{-4}) \\ \phi_1 &= \frac{\phi_1^0}{r^2} + O(r^{-3}) \\ \phi_{2B} &= \frac{\phi_2^0}{r} + O(r^{-2}). \end{aligned} \quad (5)$$

The vector field l^a is the tangent field to the null geodesic generators of the null cones \mathfrak{C}_0 . At \mathfrak{J}^+ , n^b is the tangent field to the null generators of \mathfrak{J}^+ while

(m^a, \bar{m}^b) are (the complex conjugate pair) tangent to the two surface, S^2 , the intersection of \mathfrak{C}_0 with \mathfrak{I}^+ .

The r independent quantities $(\phi_0^0, \phi_1^0, \phi_2^0, \dots)$ are functions ‘living’ on \mathfrak{I}^+ , i.e., functions of the retarded time, u , (the light-cone cuts of \mathfrak{I}^+) and $(\zeta, \bar{\zeta})$, the complex stereographic coordinates labeling the, S^2 , generators of \mathfrak{I}^+ . The components of their spherical harmonic decomposition,

$$\phi_0^0 = \phi_{0i}^0 Y_{1i}^1 + \phi_{0ij}^0 Y_{2ij}^1 + \dots, \quad (6)$$

$$\phi_1^0 = Q + \phi_{1i}^0 Y_{1i}^0 + \phi_{1ij}^0 Y_{2ij}^0 + \dots, \quad (7)$$

$$\phi_2^0 = \phi_{2i}^0 Y_{1i}^{-1} + \phi_{2ij}^0 Y_{2ij}^{-1} + \dots, \quad (8)$$

are the asymptotically defined multipole moments and their time derivatives.

For example, the $l = 0$ harmonic component of ϕ_1^0 is proportional to the total source charge. For us the important quantity is ϕ_{0i}^0 , the $l = 1$ component of ϕ_0^0 : ϕ_{0i}^0 is proportional to the (*asymptotically defined*) complex dipole moment, $\vec{D}_C = \vec{D}_E + i\vec{D}_M$, where \vec{D}_M is the magnetic dipole moment.

The problem now is: how does the \vec{D}_C transform under an origin shift to an arbitrary world-line? With an origin shift there will be new light-cones and a new null vector field, l^{*a} , obtained from the old one, l^a , by a null rotation at \mathfrak{I}^+ . This can be expressed explicitly by[9, 10]

$$\begin{aligned} l^* &= l + \frac{L}{r} \bar{m} + \frac{\bar{L}}{r} m + O(r^{-2}) \\ m^* &= m + O(r^{-1}) \\ n^* &= n \end{aligned} \quad (9)$$

where $L = L(u, \zeta, \bar{\zeta})$ is a stereographic angle field given on \mathfrak{I}^+ (still to be described) that determines the new null geodesic field, l^* .

The transformation law for ϕ_{0i}^0 (given only approximately for small origin shifts[8]) is[6]

$$\phi_{0i}^{*0} = (\phi_0^0 - 2L\phi_1^0 + \dots)_i \quad (10)$$

If we are given a Minkowski space world-line, $x^a = \xi^a(s)$, for the apex of the new light cones, then $L = L(u, \zeta, \bar{\zeta})$ is given in the parametric form

$$\begin{aligned} L(u, \zeta, \bar{\zeta}) &= \xi^a(s) m_a(\zeta, \bar{\zeta}), \\ u &= \xi^a(s) l_a(\zeta, \bar{\zeta}), \end{aligned} \quad (11)$$

with

$$l_a(\zeta, \bar{\zeta}) = \frac{\sqrt{2}}{2} \left(1, \frac{\zeta + \bar{\zeta}}{1 + \zeta \bar{\zeta}}, -i \frac{\zeta - \bar{\zeta}}{1 + \zeta \bar{\zeta}}, \frac{-1 + \zeta \bar{\zeta}}{1 + \zeta \bar{\zeta}} \right), \quad (12)$$

$$m_a(\zeta, \bar{\zeta}) = (0, Y_{1i}^1(\zeta, \bar{\zeta})) = \frac{\sqrt{2}}{2P} (0, 1 - \bar{\zeta}^2, -i(1 + \bar{\zeta}^2), 2\bar{\zeta}), \quad (13)$$

By the appropriate choice of $\xi^a(s)$, from Eq.(10), with the use of Eq.(11), one can force the real part of ϕ_{0i}^{*0} to vanish, thereby making $x^a = \xi^a(s)$ the

center of charge. If however we generalized the choice of $L(u, \zeta, \bar{\zeta})$ and allowed it to be defined parametrically by

$$\begin{aligned} L(u, \zeta, \bar{\zeta}) &= \xi_C^a(\tau) m_a(\zeta, \bar{\zeta}), \\ u &= \xi_C^a(\tau) l_a(\zeta, \bar{\zeta}), \end{aligned} \quad (14)$$

where $z^a = \xi_C^a$ is a complex analytic world in complex Minkowski space, then by setting $\phi_{0i}^{*0} = 0$, in Eq.(10) the complex world-line is determined. This complex curve (which is purely formal) defines the *complex center of charge*. Using this "curve" as the origin both the electric and magnetic dipoles vanish.

Theorem[11] - copy from home

3 The Complex Center of Mass

For asymptotically flat Einstein-Maxwell space-times the situation is totally analogous: the shear-free null geodesics originating from light-cones from world-lines (real or complex) are replaced by (regular) asymptotically shear-free null geodesic congruences generated by a complex world-line[11], in the space of the complex Poincare translation subgroup of the BMS group. The Maxwell asymptotic dipole transforms exactly as in the flat space case, i.e., as in Eq.(10) with however a slight change in the parametric description of the function $L(u, \zeta, \bar{\zeta})$:

$$L = \xi^i(\tau) Y_{1i}^1(\zeta, \bar{\zeta}) - 6\xi^{ij}(\tau) Y_{2ij}^1(\zeta, \bar{\zeta}), \quad (15)$$

$$u = \frac{1}{\sqrt{2}} \xi^0(\tau) - \frac{1}{2} \xi^i(\tau) Y_{1i}^0(\zeta, \bar{\zeta}) + \xi^{ij}(\tau) Y_{2ij}^0(\zeta, \bar{\zeta}) + \dots, \quad (16)$$

where the extra terms come from the existence of a non-vanishing Bondi shear, given up to $l = 2$ terms, by

$$\sigma = 24\xi^{ij}(u) Y_{2ij}^2 + \dots \quad (17)$$

Turning the gravitational behavior, the relevant (for us) tetrad components of the Weyl tensor[10, 6]

$$\begin{aligned} \psi_1 &= -C_{abcd} l^a m^b l^c m^d \\ \psi_2 &= -C_{abcd} \bar{m}^a n^b l^c m^d \end{aligned}$$

have the asymptotic form (the peeling theorem)

$$\begin{aligned} \psi_1 &= \frac{\psi_1^0(u, \zeta, \bar{\zeta})}{r^4} + O(r^{-5}) \\ \psi_2 &= \frac{\psi_2^0(u, \zeta, \bar{\zeta})}{r^3} + O(r^{-4}) \end{aligned}$$

The leading terms have the harmonic expansion:

$$\psi_2^0 = \Upsilon + \psi_{2i}^0 Y_{1i}^0 + \psi_{2ij}^0 Y_{2ij}^0 + \dots \quad (18)$$

$$\psi_1^0 = \psi_{1i}^0 Y_{1i}^1 + \psi_{1ij}^0 Y_{2ij}^1 + \dots \quad (19)$$

The mass aspect, defined by

$$\Psi = \psi_2^0 + \eth^2 \bar{\sigma} + \sigma(\bar{\sigma}),$$

is *real* and has the expansion

$$\Psi = \bar{\Psi} = \Psi^0 + \Psi^i Y_{1i}^0 + \Psi^{ij} Y_{2ij}^0 + \dots$$

The Bondi mass and linear momentum (four-momentum) is obtained from the $l = (0, 1)$ harmonic components of Ψ by:

$$\Psi^0 = -\frac{2\sqrt{2}G}{c^2}M \quad (20)$$

$$\Psi^i = -\frac{6G}{c^3}P^i. \quad (21)$$

The *complex gravitational dipole moment* (roughly mass dipole plus i times angular momentum) is identified as being proportional to the $l = 1$ harmonic of ψ_1^0 , i.e., as ψ_{1i}^0 . (Though many authors add further terms to ψ_{1i}^0 for this identification, to our order of approximation they all agree with our identification[6].)

The transformation (to second order) of ψ_{1i}^0 to an arbitrary (complex) world-line, analogous to Eq.(10), using Eq.(15), is[6]

$$\psi_{1i}^{*0} = (\psi_1^0 - 3L\psi_2^0 + \dots)_i. \quad (22)$$

Setting $\psi_{1i}^{*0} = 0$, thereby defining the complex center of mass, $\xi^i(u)$, yields after a lengthy calculation,

$$\psi_{1i}^0 = -\frac{6\sqrt{2}G}{c^2}M[\xi^i(w) + i\frac{1}{2}\epsilon_{kji}v^k\xi^j] + G^i, \quad (23)$$

where G^i is a known non-linear function of quadrupole terms.

At this point we make our only assumption on the physical system being considered. We saw that we could determine a complex center of charge or a complex gravitational center of mass by setting either φ_{0i}^{*0} or ψ_{1i}^{*0} to zero. *We now assume, for the rest of this work, that the two complex world-lines coincide.* Aside from taking $Q \neq 0$, there are no other conditions on the internal structure of our source (particle).

We now turn to the dynamics, which are contained in the asymptotic Bianchi identities. They can be written:

$$(\psi_1^0)' = -\partial\Psi + \partial\sigma(\bar{\sigma})' + 3\sigma\partial(\bar{\sigma})' + \partial^3\bar{\sigma} + 2k\phi_1^0\bar{\phi}_2^0 \quad (24)$$

$$\Psi' = \sigma'\bar{\sigma}' + k\phi_2^0\bar{\phi}_2^0 \quad (25)$$

$$k = 2Gc^{-4} \quad (26)$$

From these two equations, (24) and (25), we extract the equations of motion with the radiation reaction term. Rather than going thru the details (long with rather unattractive calculations) we will describe what we did in words and then give the results. First we point out that we change the Bondi u -variable to $w = \sqrt{2}uc^{-1}$, w being the retarded time. Derivatives with respect to w are denoted by prime, $(')$.

We first extract from Eq.(24) its $l = 1$ part and then decompose it into its real and imaginary parts. This yields two results: the imaginary part determines the dynamics of the total angular momentum, i.e., the conservation of angular momentum. Other than remarking that we identify $S^i = Mc\xi_I^i$ as the intrinsic spin (with ξ_I^i the imaginary part of ξ^i), this is not our interest here and will not be discussed any further. The real $l = 1$ part can be solved for the linear momentum P^i that was sitting in the $l = 1$ part of Ψ :

$$P^k = Mv_R^k - \frac{2Q^2}{3c^3}v_R^{k'} + W^k. \quad (27)$$

This is a major result that come from our identification of the complex centers of mass and charge. First of all we see *kinematic* expressions for the Bondi 3-momentum, the mv term and then the *radiation reaction contribution to the momentum*. The W contains further kinematic terms involving spin and quadrupole interactions that are known but not displayed here[6].

Extracting the $l = (0, 1)$ harmonics from Eq.(25) yields the Bondi mass and momentum loss equations:

$$M' = -\frac{G}{5c^7}(Q_{Mass}^{ij'''}Q_{Mass}^{ij'''} + Q_{Spin}^{ij'''}Q_{Spin}^{ij'''}) - \frac{2Q^2}{3c^5}(v_R^{i'}v_R^{i'} + v_I^{i'}v_I^{i'}) \quad (28)$$

$$- \frac{1}{180c^7}(D_E^{ij'''}D_E^{ij'''} + D_M^{ij'''}D_M^{ij'''})$$

$$P^{k'} = F^k \equiv \frac{2G}{15c^6}(Q_{Spin}^{lj'''}Q_{Mass}^{ij'''} - Q_{Mass}^{lj'''}Q_{Spin}^{ij'''})\epsilon_{ilk} - \frac{Q^2}{3c^4}(v_I^{l'}v_R^{i'} - v_R^{l'}v_I^{i'})\epsilon_{ilk} \quad (29)$$

$$+ \frac{Q}{15c^5}(v_R^{j'}D_E^{jk'''} + v_I^{j'}D_M^{jk'''}) + \frac{1}{540c^6}(D_E^{lj'''}D_M^{ij'''} - D_M^{lj'''}D_E^{ij'''})\epsilon_{ilk}$$

with the mass and spin quadrupoles related to the ξ^{ij} by

$$\xi^{ij} = (\xi_R^{ij} + i\xi_I^{ij}) = \frac{G}{12\sqrt{2}c^4}(Q_{Mass}^{ij''} + iQ_{Spin}^{ij''}). \quad (30)$$

The mass loss equation is thus exactly the usual quadrupole energy loss plus the classical dipole and quadrupole electromagnetic energy loss.

It is however Eq.(29) that is of most interest to us. By substituting the kinematic expression for the momentum, Eq.(27) into Eq.(29) we obtain our generalized Abraham-Lorentz equations of motion:

$$Mv_R^{k'} + v_R^k M' - \frac{2Q^2}{3c^3} v_R^{k''} + R^k = F^k. \quad (31)$$

Note that though it is similar to the Abraham-Lorentz equations there are many differences that are hidden in the known but complicated expressions for M' , R^k and F^k . The F^k is the Bondi recoil (or rocket) force due to the momentum loss, while R^k can be considered to be a gravitational radiation reaction force depending on internal degrees of freedom spin and quadrupole moments. M' has exactly the classical dipole energy loss terms plus two additional terms. Though it is very hard to directly see if the solutions to Eq.(31) are well behaved, in the conclusion we will discuss this issue in more general terms.

4 Conclusions

We have considered the situation of a gravitating - electromagnetic source of compact support viewed from future null infinity. The only restriction made on the distributions is that the total charge is non-vanishing and that the *complex asymptotic electromagnetic dipole moment* be proportional the *complex gravitational dipole moment* so that the complex centers of charge and mass coincide. Though it is not clear how severe this condition is, it certainly is a serious restriction. It has been shown that for this situation the gyromagnetic ration, the ratio of the spin-angular momentum to the magnetic moment is that of Dirac's, namely $g = 2$. We showed that in a manner completely analogous to the flat space Maxwell case, one could determine the transformation laws for the two dipole moments and thereby go to the center of mass/charge, determining a unique complex world-line. Then, using the Bianchi Identities, which play the role of dynamical equations, we were able to give kinematic significance to the Bondi linear momentum, in the sense that we had

$$\vec{P} = M\vec{v} - \frac{2Q^2}{3c^3} \vec{v}' + \dots \quad (32)$$

From the Bondi momentum loss equation it immediately followed that we had a generalized version of the Abraham-Lorentz equations of motion for an isolated massive charged particle. It should be emphasized that the quadrupole quantity, $\xi^{ij}(u)$, is arbitrary and in most cases it is taken as non-vanishing in a finite interval so that the gravitational radiation exists also in a finite interval. If however *the motion*, from Eq.(31), *is unstable*, the particle acceleration will be unbounded and there will be an infinite energy loss via the *electromagnetic dipole radiation*. This would be a physically unacceptable situation, indicating that something is seriously wrong with the Einstein-Maxwell equations.

The question then is does the general relativity (gravitational) contributions to the equations of motion stabilize the equations. Though we do not see any immediate prospects for a direct proof, we make a few comments. Looking at Eq.(31), we see that the term $M'v$ has the same form as in the Abraham-Lorentz equation but now is more negative because of the extra radiation terms and has the correct sign to try to stabilize the motion. Whether or not it does stabilize is an open question. If not, perhaps higher order terms that have been left out in our approximations might succeed. And finally, it is known that the vacuum Einstein equations are stable in the neighborhood of Minkowski space. If the same were true of the Einstein-Maxwell equations with compact sources, that would constitute a proof that our physical situation was indeed stable and the run-away behavior was prevented by the inclusion of classical general relativity. Unfortunately, this is a difficult question and, to our understanding, the answer is unknown. It would be surprising if it turned out that the asymptotic Einstein-Maxwell equations were unstable.

5 Acknowledgements

G.S.O. acknowledges the financial support from CONACYT and Sistema Nacional de Investigadores (SNI-México). C.K. thanks CONICET and SECYTUNC for support.

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